

Lubrication Flow in a Narrow Gap

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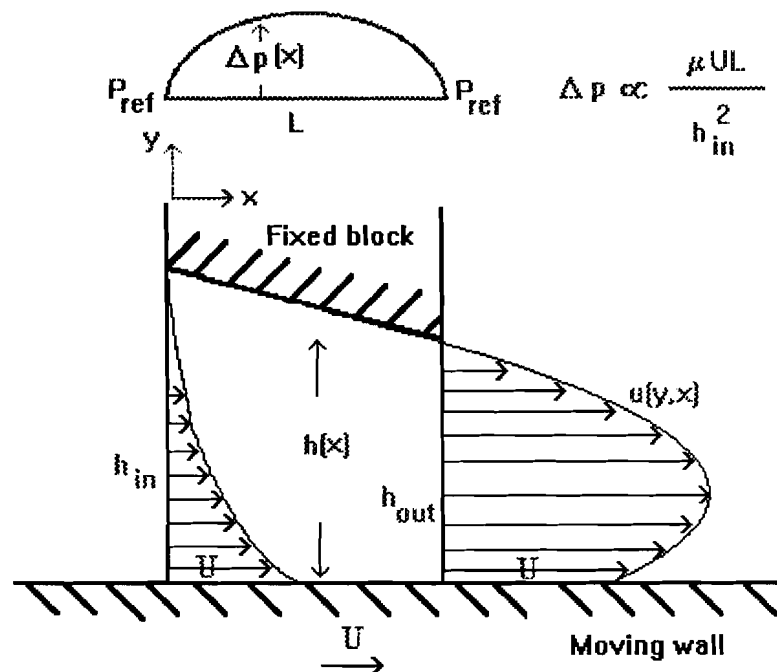
Fluid flow in a narrow gap is a very characteristic problem in Fluid Mechanics, or more specifically in lubrication theory, developed by Reynolds in 1886. When two bodies move relative to each other, the friction is reduced if a viscous fluid (lubricant) with viscosity μ is allowed to move through a narrow but variable gap between them. An idealization of lubrication problem is the slipper-pad bearing shown in Fig. 1. The bottom wall moves relative to the upper block at a velocity U and creates a Couette flow in the gap. It is important that the gap is very narrow ($h \ll L$) and non-uniform ($h(x) \neq \text{const}$). The pressure distribution $p(x)$ along the gap ($x = 0 \dots L$) is described by the following governing, second-order differential equation (see for example [1]) :

$$\frac{d}{dx} \left(h(x)^3 \cdot \frac{d}{dx} p(x) \right) = 6 \cdot \mu \cdot U \cdot \frac{d}{dx} h(x)$$

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Figure 1:
Slipper-pad bearing.

We will calculate the excess pressure distribution, Δp .



Problem Statement:

For a given (arbitrary) gap profile, i.e. some equation $h(x)$, solve for the pressure distribution profile $p(x)$ using the above differential equation. To see the effect of different parameters on the result, calculate the excess pressure distribution $\Delta p(x) = p(x) - p_{ref}$ (where the reference pressure p_{ref} is a known constant), for a decreasing, increasing and uniform gap, and for different gap types, with various viscosities and velocities. The physics and/or trends of the problem will be better understood if the user "plays" interactively with different input values.

Input variables (arbitrary and in any system of units):

The bottom wall
velocity:

$$U := 20 \cdot \frac{\text{m}}{\text{sec}}$$

The fluid (lubricant)
viscosity:

$$\mu := 0.5 \cdot \text{Pa} \cdot \text{sec}$$

Gap dimensions and profile:

Height of the gap at
the inlet ($h_{in} < 0.01L$
recommended):

$$h_{in} := 0.1 \cdot \text{in}$$

Height of the gap at the
outlet ($h_{out} < h_{in}$
recommended):

$$h_{out} := 0.05 \cdot \text{in}$$

Gap profile polynomial order:
(e.g. $n = 0, 1, 2, 3, \dots$ implies
respectively a uniform, linear,
quadratic, cubic profile, etc.)

$$n := 2$$

Length of the gap
in flow direction:

$$L := 5 \cdot \text{in}$$

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The gap profile equation (may be arbitrary; we chose this type for convenience):

$$h(x) := h_{out} + (h_{in} - h_{out}) \cdot \left(1 - \frac{x}{L}\right)^n$$

Solution:

The governing second-order differential equation may be expressed in more universal, dimensionless form if its dimensional variables are normalized using the above input variables.

First, the dimensionless gap length in the flow direction should be defined as a range variable for future calculations.

$$\xi_x = \frac{x}{L} \quad \text{so that} \quad \xi := 0, \frac{1}{20} \dots 1 \quad \text{or} \quad x(\xi) := \xi \cdot L$$

The dimensionless gap height is then given by

$$\eta(\xi) := \frac{h(x(\xi))}{h_{in}}$$

And the dimensionless pressure is

$$\pi(\xi) = \frac{\Delta p(x) \cdot h_{in}^2}{\mu \cdot U \cdot L}$$

where the function for dimensional gap excess pressure is $\Delta p(x) = p(x) - p_{ref}$

The reference pressure, p_{ref} , is the environmental pressure (outside the gap), and is usually 0 psi (gauge) atmospheric pressure. The boundary conditions at the inlet and outlet are therefore

$$\Delta p(0) = \Delta p(L) = 0 \quad \text{or} \quad \pi(0) = \pi(1) = 0$$

Finally, with the above substitutions, we obtain the governing (Reynolds') equation in dimensionless form:

$$\frac{d}{d\xi} \left(\eta(\xi)^3 \frac{d}{d\xi} \pi(\xi) \right) = 6 \cdot \frac{d}{d\xi} \eta(\xi)$$

Integrating both sides of this equation with respect to ξ , dividing by $\eta(\xi)^3$, and integrating one more time with respect to ξ gives the final solution for the dimensionless pressure $\pi(\xi)$.

$$\pi(\xi) = \int_0^\xi \frac{\int_0^\xi 6 \cdot \frac{d}{d\xi} \eta(\xi) d\xi + C}{\eta(\xi)^3} d\xi$$

The interior integral can easily be solved analytically without losing any generality. We may express the dimensionless pressure, π , as a function of dimensionless length, ξ and an integration constant, C .

$$\pi(\xi, C) := \int_0^\xi \frac{C + 6 \cdot (\eta(\xi) - \eta(0))}{\eta(\xi)^3} d\xi \quad \text{Note that} \quad C = \eta(0)^3 \cdot \frac{d}{d\xi} \pi(0)$$

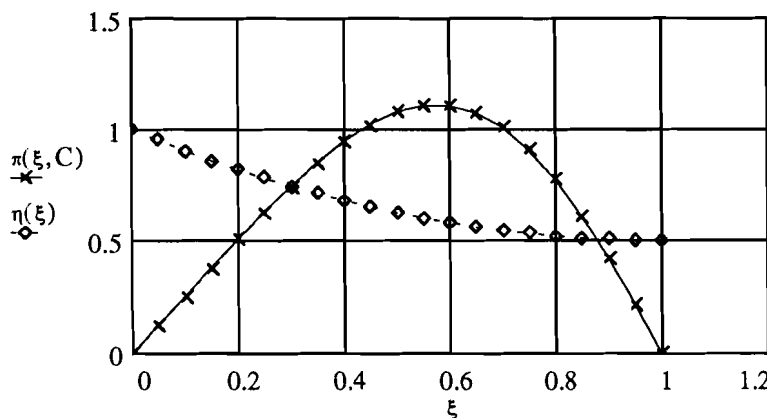
This is a common form of a boundary-value problem, which requires an iterative numerical solution. In order to satisfy the known boundary condition, $\pi(1, C) = 0$, the constant of integration can be calculated using the root finder.

First, make an initial guess for C : $C := 1$

Then, calculate the approximate value: $C := \text{root}(\pi(1, C), C)$

$C = 2.459$

Now that the integration constant C is determined, the universal, dimensionless pressure π and the gap profiles η may be easily plotted as a function of dimensionless gap length, ξ .

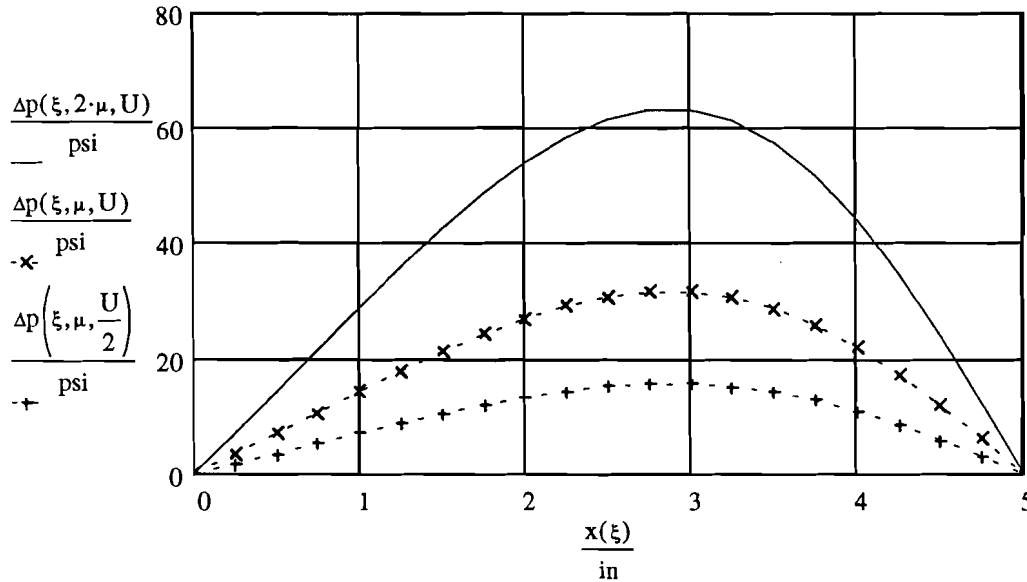


The pressure profile peaks just past the center of the gap, where the gap profile narrows to a waist.

Corresponding dimensional variables, like gap excess pressure Δp or gap height h as function of gap length x may be easily calculated and/or plotted.

$$\Delta p(\xi, \mu, U) := \frac{\pi(\xi, C) \cdot \mu \cdot U \cdot L}{h_{in}^2}$$

The following graph compares pressure profiles for several combinations of input variables, showing the results of increasing viscosity or decreasing velocity from the original conditions.



It is interesting to notice that the solution for the dimensionless (normalized) pressure profile is universal, i.e. independent on fluid type or velocity. The fluid viscosity and relative velocity are only used for, and do influence, the dimensional pressure as seen above, but the basic shape of the pressure distribution will depend on the gap profile alone.

Verifying the Solution:

Any solution looks good on paper, but now we should check that it makes sense. First, verify the boundary conditions used to solve the problem.

$$\Delta p(0, \mu, U) = 0 \text{ psi} \quad \text{and}$$

$$\Delta p(1, \mu, U) = 4.284 \cdot 10^{-11} \text{ psi}$$

which is nearly zero. The discrepancy is the result of the numerical process of root finding and integration.

Next, note that the bearing's load per unit area has to be smaller than the average gap excess pressure, Δp_{ave} . If this condition is not met, the lubricant will not prevent the bearing surfaces from rubbing against one another. This number can also be extracted using these calculations.

$$\pi_{ave} := \int_0^1 \pi(\xi, C) d\xi \quad \pi_{ave} = 0.689 \quad \Delta p_{ave} := \frac{\pi_{ave} \cdot \mu \cdot U \cdot L}{h_{in}^2} \quad \Delta p_{ave} = 19.662 \cdot \text{psi}$$

We may calculate the pressure extremes (maximum or minimum) within the gap by finding its location (ξ_{ext} or x_{ext}) where the first derivative of the pressure profile is zero.

$$\begin{aligned} \text{Guess:} \quad \xi_{ini} &:= 0.5 & \xi_{ext} &:= \text{root}\left(\frac{d}{d\xi} \pi(\xi_{ini}, C), \xi_{ini}\right) \\ \xi_{ext} &= 0.575 & \pi_{ext} &:= \pi(\xi_{ext}, C) & \pi_{ext} &= 1.11 \end{aligned}$$

The corresponding dimensional, extreme excess pressure, and its location, are

$$\begin{aligned} \Delta p_{ext} &:= \Delta p(\xi_{ext}, \mu, U) & \Delta p_{ext} &= 31.69 \cdot \text{psi} \\ x_{ext} &:= x(\xi_{ext}) & x_{ext} &= 2.877 \cdot \text{in} \end{aligned}$$

The above procedure will be applicable for any input gap profile equation, $h(\mathbf{x})$. The governing dimensionless differential equation will always maintain the same closed form symbolically in terms of $\pi(\xi)$ and $\eta(\xi)$. You may conveniently investigate the influence of different input variables on the results. For example, to find the inverse solution (reversed flow), simply swap the values for h_{in} and h_{out} in the beginning of the document. You should find that this produces undesirable, negative excess pressure, i.e. vacuum. For a uniform gap profile, $n = 0$, the excess pressure Δp will be zero everywhere (again undesirable). Therefore, for effective lubrication flow, it is important to have a decreasing narrow gap in the direction of flow to obtain enough excess pressure differential to prevent two bearing surfaces to rub (touch) each other under their load, thereby reducing the friction. You are encouraged to experiment with other configurations and see the influence on the results.

For an example of the type of calculations to be done when there is no lubrication between bearings, see the article in this issue on [Hertzian stress analysis](#).

Reference

[1] F. M. White, *Viscous Fluid Flow, 2nd Ed.*, McGraw-Hill Pub. Co., (New York, 1991).

MathSoft

$\Sigma + \sqrt{\quad} - = \times f \div \delta$

July 14, 1993

Dr. Milivoje Kostic
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Dear Dr. Kostic:

Thank you for submitting your article on **Lubrication Flow through a Narrow Gap** to *Applied Mathcad*. I am delighted to tell you that your article has been reviewed and accepted for publication in our journal. It will appear in the Mathcad in Education section of the Fall issue: Volume 2, Number 3, 1993.

It has been a pleasure to work with a professional both in the area of Mechanical Engineering, and in the area of Engineering Education. I am impressed with your command of the subject and your commitment to your students.

Good luck in endeavors, and please let me know if you have other submissions I may consider for future issues.

Sincerely,



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