Ch.5: Uncertainty/Error Analysis

- Introduction
- Bias and Precision Errors/Uncertainties
- Different Instrument Errors
- RSS-Errors Summing
- Error Sources
- Sensitivity or Dependency Rate
- Error Propagation (Combined Uncertainty)
- Design-stage Uncertainty
  (Instrument errors only)
- Advanced-stage Uncertainty
  (Instrument and measurement errors)
- Multiple-measurement Uncertainty
- Error Summation/Propagation
  (Expanded Combined Uncertainty)
- Problem 5-30

Ch.5: Uncertainty/Error Analysis

Errors (**e = x-x’** @ **$x\text{-}x_{avg}$= d**, also B, P, S≅σ;
do NOT be confused, see NIST Guide):

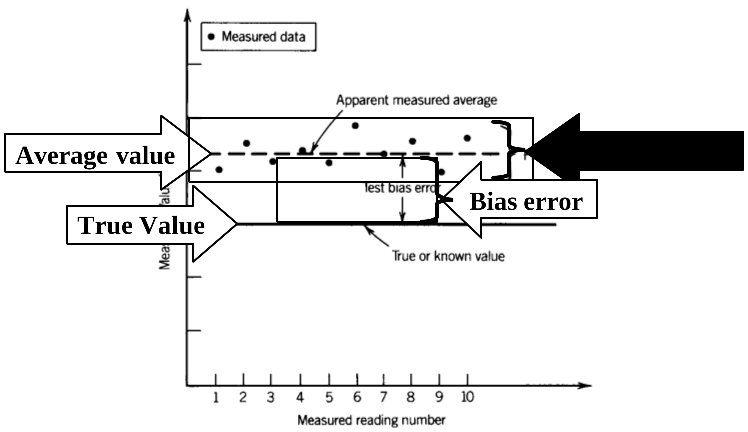
Bias (B), Precision (P), also Standard (S or s**)**

Uncertainty (u) is the range of errors (e, B, P, S) at corresponding Probability (%P)

Remember: $u = d_{\%P} = t_{\nu,\%P}S$ (@ %P); z=t=d/S

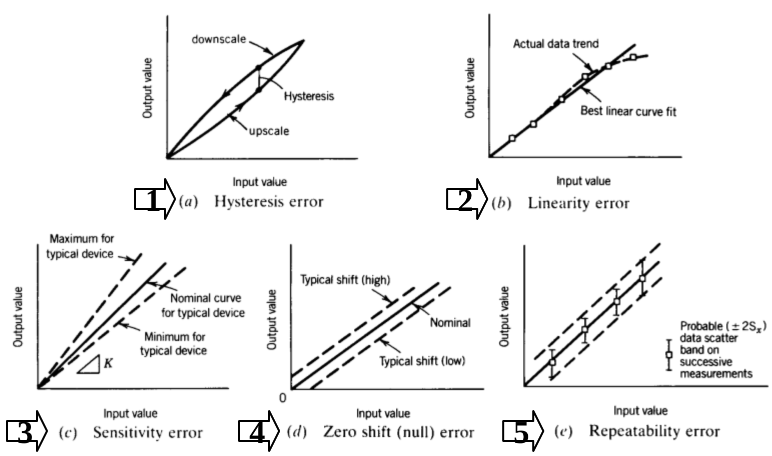
Bias and Precision Errors/Uncertainties

FIGURE 1.9 Effects of precision and bias errors on calibration readings.



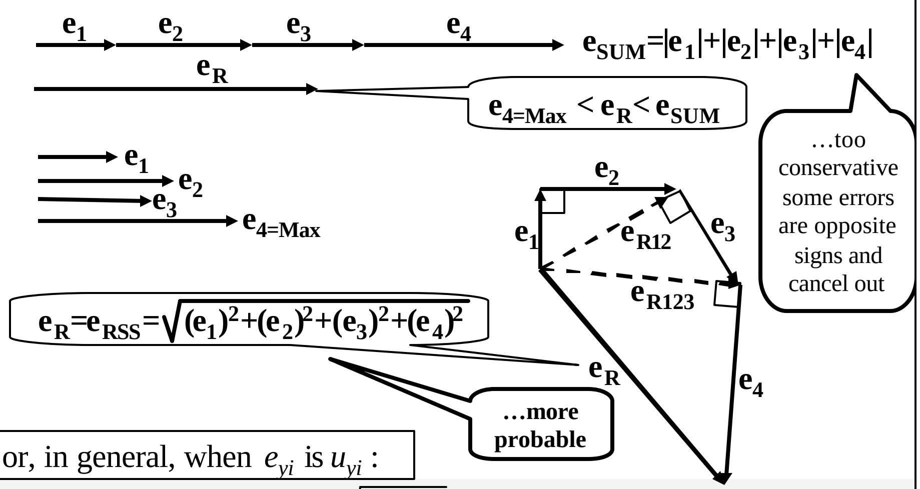
Different Instrument Errors

FIGURE 1.10 Examples of elements of instrument error. (a) Hysteresis error. (b) Linearity error. (c) Sensitivity error. (d) Zero shift (null) error. (e) Repeatability error.



$$u_y = \sqrt{u_{y1}^2 + u_{y2}^2 + ... + u_{yL}^2} = \sqrt{\sum_{i=1}^{L} u_{yi}^2}$$; i.e. RSS (Root – Sum – Square) procedure

RSS-Errors Summing



…or, in general, when e_{yi} is u_{yi} :

$$u_y = \sqrt{u_{y1}^2 + u_{y2}^2 + ... + u_{yL}^2} = \sqrt{\sum_{i=1}^{L} u_{yi}^2}$$; i.e. RSS (Root – Sum – Square) procedure

Error Sources

1. Calibration Error Source Group

Table 5.1 - (B or P)$_{1j}$

2. Data Acquisition Error Source Group

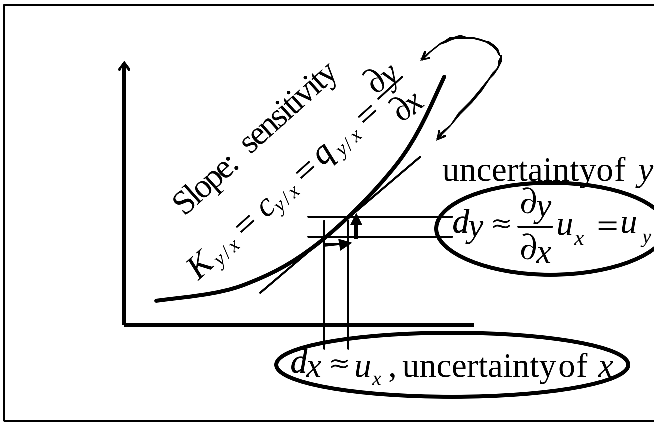
Table 5.2 - (B or P)$_{2j}$

3. Data Reduction Error Source Group

Table 5.3 - (B or P)$_{3j}$

It is not important which group an error is assigned to,
as long as it is accounted for.
The groups and their items are for convenience only.

Sensitivity or Dependency Rate



If one variable "y" depends on another "x," then a small change of x, i.e $dx \approx u_x$ (error, uncertainty) will propagate as error of y, i.e. $dy \approx u_y$.

Using the partial derivative, i.e. the rate of dependancy, or sensitivity:

$$dy \approx \frac{\partial y}{\partial x} dx \approx \frac{\partial y}{\partial x} u_x = (q_{y/x}) u_x = u_y \text{ ; where sensitivity } K_{y/x} = c_{y/x} = q_{y/x} = \frac{\partial y}{\partial x}$$

Error Propagation (Combined Uncertainty)

If one variable y depends on another x, then a small change of x, i.e $dx \approx u_x$ (error, uncertaint y) will propagate as error of y, i.e. $dy \approx u_y$:

$$dy \approx \frac{\partial y}{\partial x} dx \approx \frac{\partial y}{\partial x} u_x = (q_{y/x}) u_x = u_y \text{ ; where sensitivit y } q_{y/x} = \frac{\partial y}{\partial x}$$

For a multyfunct ion variable $y = y(x_1, x_2, \dots x_i, \dots x_L)$:

$$u_y = \sqrt{\sum_{i=1}^{L} \left(q_{y/xi} u_{xi}\right)^2} = \sqrt{\sum_{i=1}^{L} u_{yi}^2} \text{ ; i.e. RSS (Root – Sum – Square) procedure}$$

$$\text{where } u_{yi} = \begin{cases} u_{yi} \text{ a known elemental error} \\ \text{or } u_{yi} = \left(\frac{\partial y}{\partial x_i}\right) u_{xi} = (q_{yi/x}) u_{xi} \end{cases}$$

Design-stage Uncertainty

(Instrument errors only)

Design - stage error/uncertainty

$$u_d = \sqrt{u_0^2 + u_c^2}$$

Interpolation error

$u_0 = \pm \frac{1}{2}$ resolution

Instrument error

u_c (*by calibration*)

Advanced-stage Uncertainty

(Instrument and measurement errors)

N$^{\text{th}}$ order uncertainty

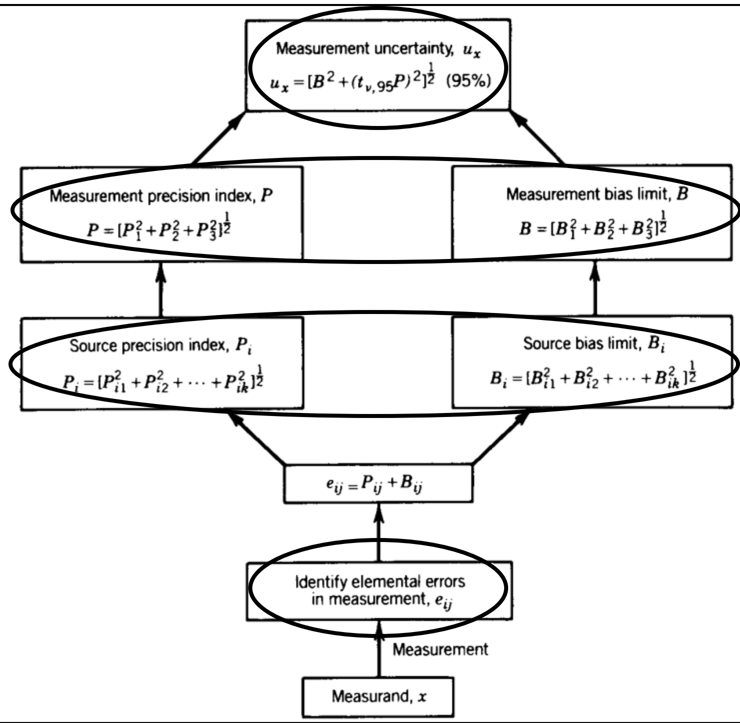
$$u_N = \sqrt{u_d^2 + \sum_{i=1}^{N} u_i^2}$$

Zero - and design order uncertainties

$u_0 = \pm \frac{1}{2}$ resolution; $u_d = \sqrt{u_0^2 + u_c^2}$

First - order uncertainty

$u_1 \geq u_0, u_d$



Multiple-measurement Uncertainty

Error Summation/Propagation

(Expanded Combined Uncertainty)

$$u_R = \sqrt{B_R^{\,2} + (t_{n_R,\%P} P_R)^2}\text{, where :}$$

$$B_R = \sqrt{\sum_i B_i^{\,2}} \quad \text{and} \quad P_R = \sqrt{\sum_i P_i^{\,2}} \quad \text{; also :}$$

$$n_R = \frac{\left(\sum_i P_i^{\,2}\right)^2}{\sum_i \frac{P_i^{\,4}}{n_i}} \;; \quad n_i = N_i - 1,\ \#\text{ of degree of freedom}$$

Note, it could be $P_i = P_{yi} = q_{yi/xi} P_{xi}$

or, it could be $B_i = B_{yi} = q_{yi/xi} B_{xi}$

Problem 5-30

MEE 390/490 Problem 5.30, p.214

A temperature measurement system (termocouple for example) is calibrated against a standard system (mercury thermometer for example) certified to an uncertainty of +-0.05C at 95%. The system sensor is immersed alongside the standard within a temperature bath so that the two are separated by about 10 mm. The temperature uniformity of the bath is estimated at about 5C/m. The temperature system sensor is connected to a readout (multimeter for example) that indicates the temperature in terms of voltage. The following are calibration results between the temperature indicated by the standard **T[C]=y** and the indicated voltage **E[mV]=x**, i.e.:

$$x := \begin{pmatrix} 0.004 \\ 0.399 \\ 0.771 \\ 1.624 \\ 2.147 \\ 4.121 \end{pmatrix} \qquad y := \begin{pmatrix} 0.1 \\ 10.2 \\ 19.5 \\ 40.5 \\ 51.2 \\ 99.6 \end{pmatrix}$$

NOTE:
x = E[mV] =measured sensor's output electromotive force (voltage)
y = T[C] =measured temperature of the temperature standard

a) Compute the calibration curve fit. b) Estimate the uncertainty in using the output from the temperature measurement system for temperature measurements

Problem 5-30 (Cont.)

Solution:

Evaluate the number of data points, or the size-length of vector **x** or **y**:

$N := \text{length}(x) \qquad N = 6$ Check (see) couple of data

Then, data range variable will be: $x_0 = 4\times 10^{-3}$ $\quad x_{N-1} = 4.121$

$i := 0..\, N-1$ $\quad y_0 = 0.1$ $\quad y_{N-1} = 99.6$

Compute sample statistics using the buit-in MathCAD functions:

$\text{mean}(x) = 1.511 \qquad \text{mean}(y) = 36.85$

$\text{var}(x) = 1.881 \qquad \text{var}(y) = 1.086\times 10^{3}$

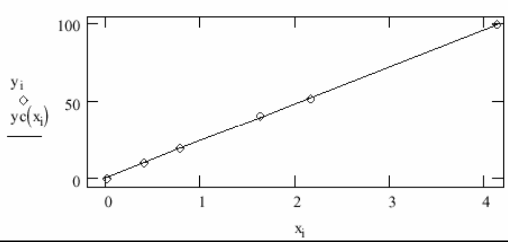
$\text{stdev}(x) = 1.371 \qquad \text{stdev}(y) = 32.961$

$\text{corr}(x,y) = 0.99983$

$b := \text{intercept}(x,y) \qquad b = 0.54$

$K := \text{slope}(x,y) \qquad K = 24.03$

Finally, plot the data and the fitted line: $yc(x) := b + K\cdot x$



LEGEND:
y =measured data
yc =calculated values using fitted curve

CURVE FITTING with POLYNOMIAL of m-ORDER

by **M. Kostic** 1 by M. Kostic

Problem 5-30 (Cont.)

NOTE: The nomenclature corresponds to *"Theory and Design for Mechanical Measurements, 2nd Ed."* by Figliola and Beasley, p.149, unless given otherwise.

For a polynomial of **m**-order (m=0,1,2,...N-1, may be **arbitrary** integer within this range):

$m := 1$ ...for example

Then, the range variable and the degree of freedom will be:

$j := 0..m \qquad \nu_m := m + 1 \qquad \nu := N - \nu_m$

For a polynomial curve fit:

$$y_c = a_0 + a_1 \cdot x + a_2 \cdot x^2 + .. + a_m \cdot x^m = \sum_{i=0}^{m} a_i \cdot x^i$$

and given data set **{(x_i,y_i), i=0,1,..N-1}** of **N** data points,

we may determine coeficients $\mathbf{a_i}$ which will provide the best fit for the given conditions, using the least-square method.

Curve fit using matrix operations

Create **X** matrix ([N X (m+1)] order) first: $X^{\langle j \rangle} := \overrightarrow{\left(x^j\right)}$

and calculate (effectively) the polynomial coefficients:

$a := \left(X^T \cdot X\right)^{-1} \cdot \left(X^T \cdot y\right) \qquad a = \begin{pmatrix} 0.54 \\ 24.03 \end{pmatrix}$ ...the polynomial coefficients $\mathbf{a_i}$ $\quad b = 0.54 \quad K = 24.03$

...HOW SIMPLE!! NOTE: $a_0 = b \quad a_1 = K$

Problem 5-30 (Cont.)

...HOW SIMPLE!! NOTE: $a_0 = b \quad a_1 = K$

For "zero-order" polynomial we need a little "fix-up"

$a_0 := \text{if}\left(m = 0, a, a_0\right)$

Fitted curve (j=0...m) is:

$$y_{c_i} := \sum_j a_j \cdot \left(x_i\right)^j$$

Compute the *standard error of the fit* ***Sxy***:

$d := y - y_c \qquad D := \sum \overrightarrow{\left(d^2\right)} \qquad D_0 := \sum \overrightarrow{\left[(y - \text{mean}(y))^2\right]}$

$Sxy := \sqrt{\dfrac{D}{N - \nu_m}} \qquad Sx0 := \sqrt{\dfrac{D_0}{N - 1}}$

$Sxy = 0.746 \qquad Sx0 = 36.107$ is the Reference standard error for the "zero-order" fit, **Sx0**=**Sy** in our Text

Correlation coefficient:

$r := \sqrt{1 - \left(\dfrac{Sxy}{Sx0}\right)^2} \qquad r = 0.99979$

Problem 5-30 (Cont.)

Calculation of uncertainties:

$B_{14} := 0.05$		...due to uncertainty of the temperature standard
$B_{15} := 5 \cdot 0.01$	$B_{15} = 0.05$	...due to non-uniformity of the bath temerature (5C/m) for 10mm=0.01m separation distance
$K_{TE} := a_1$	$K_{TE} = 24.03$	...sensitivity (dT/dE) of temperature with regard to sensor's electromotive force
$B_{24} := 0.001 \cdot K_{TE}$	$B_{24} = 0.024$	...due to readout voltmetar uncertainty 0.001mV*24.03C/mv
$P_{31} := Sxy$	$P_{31} = 0.746$	...due to curve fit standard error
$\nu := 6 - \nu_m$	$\nu = 4$	$t_{4\_95} := 2.77$ ...Student-t value for 95% probability (From Table 4.4).

$u_T := \sqrt{\left[\left(B_{14}^2 + B_{15}^2 + B_{24}^2\right) + \left(t_{4\_95} \cdot P_{31}\right)^2\right]}$ $\quad \sqrt{\left(B_{14}^2 + B_{15}^2 + B_{24}^2\right)} = 0.075$ ...bias part

$t_{4\_95} \cdot P_{31} = 2.066$ ...precision part

$u_T = 2.067$ ...total temperature uncertainty @ 95% probability

$a = \begin{pmatrix} 0.54 \\ 24.03 \end{pmatrix}$ **Final result**: $T(E) = 0.54 + 24.03 \cdot E$ with $u_T = 2.067$ with 95% probability, see plot below

Now plot the data and the fitted curve to visualize the results :

If the number of data is smaller than, say Nc=60, plot the polynomial curve with at least **Nc** data

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If the number of data is smaller than, say Nc=60, plot the polynomial curve with at least **Nc** data

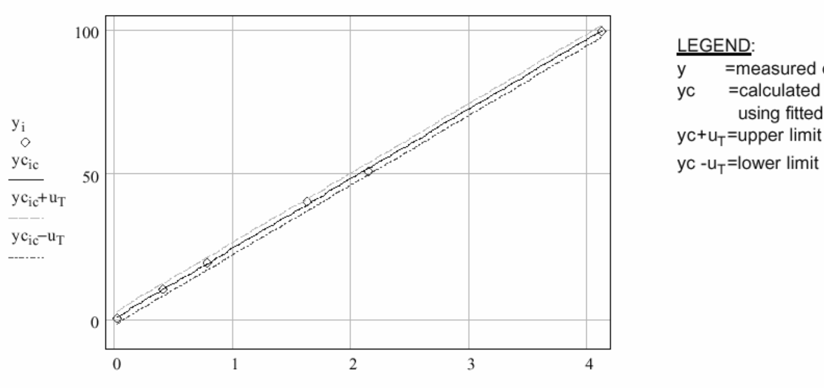
$Nc := 60$ $\quad N = 6$ ...number of data points

$Nc := \text{if}(N < Nc, Nc, N)$ $\quad dc := \dfrac{x_{N-1} - x_0}{Nc - 1}$ $\quad Nc = 60$ ...number of points to plot the curve

$ic := 0..\ Nc - 1$ $\quad m = 1$ ...order of polynomial

$xc_{ic} := x_0 + dc \cdot ic$ $\quad$ Check that these two are the same: $x_{N-1} = 4.121$ $\quad xc_{Nc-1} = 4.121$

$yc_{ic} := \sum_j a_j \cdot \left(xc_{ic}\right)^j$



Problem 5-30 (Cont.)