Polytropic Processes of Ideal Gasses: Prof. M. Kostic (Friday, April 08, 2005)

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For polytropic process:

$$Pv^n = K_n = const$$

$$d(Pv^n) = Pd(v^n) + v^n dP = Pnv^{n-1}dv + v^n dP = v^{n-1}(\underbrace{nPdv + vdP}_{=0}) = 0$$

or $nPdv + vdP = 0$ or $vdP = -nPdv$ and $\underbrace{-\int vdP}_{W_{CV}} = n\underbrace{\int Pdv}_{W}$ for polytropic process

Then:

$$d(Pv) = Pdv + vdP$$

$$\int d(Pv) = \int Pdv + \int vdP = \int Pdv + \int(-nPdv) = (1-n)\int Pdv$$

Then : $W = \int_1^2 Pdv = \frac{1}{1-n}\left[(Pv)_2 - (Pv)_1\right]$ and $W_{CV} = -\int vdP = n\underbrace{\int Pdv}_{W} = nW$

Furthermore, for constant specific heat ideal gas:

$dq = du + Pdv = C_v dT + Pdv$ (for system)

$$q = \int C_v dT + \underbrace{\int Pdv}_{W\text{ (see above)}}$$

$$q_{12} = C_v(T_2 - T_1) + \frac{1}{-n+1}(P_2v_2 - P_1v_1) = C_v(T_2 - T_1) + \frac{R}{-n+1}(T_2 - T_1)$$

$$q_{12} = C_vT_1\left(\frac{T_2}{T_1} - 1\right) + \frac{R}{-n+1}T_1\left(\frac{T_2}{T_1} - 1\right) = \left(C_v + \frac{kC_v - C_v}{-n+1}\right)T_1\left(\frac{T_2}{T_1} - 1\right)$$

$$= \underbrace{\left(\frac{n-k}{n-1}C_v\right)}_{C_n}(T_2 - T_1) = \underbrace{\left(\frac{n-k}{n-1}C_v\right)}_{C_n}T_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

where, $C_n = \frac{n-k}{n-1}C_v = \frac{n-k}{k(n-1)}C_p = \frac{n-k}{(n-1)(k-1)}R$

since $R = C_p - C_v$ and $k = \frac{C_p}{C_v}$ and $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} = \left(\frac{v_2}{v_1}\right)^{1-n}$

NOTE:

$$W_{Sys} = \underbrace{W_{CV}}_{shaft} + \underbrace{W_{CS}}_{flow} \quad \Rightarrow \quad \underbrace{\int P dv}_{W_{SYS}} = \underbrace{\int (-v) dP}_{W_{CV}} + \underbrace{\int d(Pv)}_{W_{CS}}$$

For polytropic process $Pv^n = K_n = const$ and ideal gas $Pv = RT$ were $R = C_p - C_v$ and $k = \dfrac{C_p}{C_v}$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} = \left(\frac{v_2}{v_1}\right)^{1-n}$$

$$w_{CS} = w_{flow} = \int d(Pv) = \left(P_2 v_2 - P_1 v_1\right) = R\left(T_2 - T_1\right) = RT_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

$$w_{Sys} = \int P dv = \frac{-1}{n-1} w_{flow}$$

$$w_{CV} = -\int v dP = \frac{-n}{n-1} w_{flow} = n w_{Sys}$$

$$q = \int C_n dT = \underbrace{\frac{n-k}{n-1} C_v}_{C_n}\left(T_2 - T_1\right) = \frac{C_n}{R} w_{flow} = -(n-k)\frac{C_v}{R} w_{Sys}$$

$$C_n = \frac{n-k}{n-1} C_v = \frac{n-k}{k(n-1)} C_p = \frac{n-k}{(n-1)(k-1)} R = \begin{cases} 0, & for \;\; n = k, \; adiabatic \;\; process \\ C_v, & for \;\; n = \pm\infty, \; isohoric \;\; process \\ C_p, & for \;\; n = 0, \; isobaric \;\; process \\ \pm\infty, & for \;\; n = 1, \; isothermal \;\; process \end{cases}$$

In summary, expansion work, control-volume work and heat transfer are:

$$W = \frac{1}{-n+1}(P_2v_2 - P_1v_1) = \frac{R}{-n+1}(T_2 - T_1) = \frac{RT_1}{-n+1}\left(\frac{T_2}{T_1} - 1\right) = \frac{RT_1}{-n+1}\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

$$W_{CV} = nW = \frac{n}{-n+1}(P_2v_2 - P_1v_1) = \frac{nR}{-n+1}(T_2 - T_1) = \frac{nRT_1}{-n+1}\left(\frac{T_2}{T_1} - 1\right) = \frac{nRT_1}{-n+1}\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

$$q_{12} = C_n(T_2 - T_1) = \underbrace{\left(\frac{n-k}{n-1}\cdot\frac{C_v}{R}\right)}_{C_n/R}(P_2v_2 - P_1v_1) = \underbrace{\left(\frac{n-k}{n-1}C_v\right)}_{C_n}(T_2 - T_1) = C_nT_1\left(\frac{T_2}{T_1} - 1\right) = C_nT_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

where, $C_n = \frac{n-k}{n-1}C_v = \frac{n-k}{k(n-1)}C_p = \frac{n-k}{(n-1)(k-1)}R$

Another way to derive polytropic *Pdv* work and specific heat is:

$Pv = RT$ and $P \cdot v^n = K_n = const$; thus $P = K_n v^{-n}$

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$dq = du + Pdv = C_v dT + Pdv$ (for system)

$\int Pdv = \int K_n v^{-n} dv$

$$\int_1^2 K_n v^{-n} dv = K_n \left.\frac{v^{-n+1}}{-n+1}\right|_1^2 = \left.\frac{K_n}{-n+1}\left(v_2^{-n+1} - v_1^{-n+1}\right)\right|_{K_n = P_1v_1^n = P_2v_2^n} = \frac{1}{-n+1}(P_2v_2 - P_1v_1) = \frac{RT_1}{-n+1}\left(\frac{T_2}{T_1} - 1\right)$$

Then:

$q = \int C_v dT + \int Pdv$

$$q_{12} = C_v(T_2 - T_1) + \frac{RT_1}{-n+1}\left(\frac{T_2}{T_1} - 1\right) = \left(C_v + \frac{R}{-n+1}\right)T_1\left(\frac{T_2}{T_1} - 1\right) = \underbrace{\left(\frac{n-k}{n-1}C_v\right)}_{C_n}(T_2 - T_1)$$

$$= \underbrace{\left(\frac{n-k}{n-1}C_v\right)}_{C_n}T_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

since $R = C_p - C_v$ and $k = \frac{C_p}{C_v}$ and $\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} = \left(\frac{v_2}{v_1}\right)^{1-n}$

Where, the specific heat for polytropic process is function of polytropic process index *n*, i.e.:

$$C_n = \frac{n-k}{n-1}C_v = \frac{n-k}{k(n-1)}C_p = \frac{n-k}{(n-1)(k-1)}R$$